

Pricing Restaurant Reservations: Dealing with No-Shows

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Reservation no-shows lead to wasted capacity in restaurants. In this paper we consider two remedies: punishing no-shows by charging fees and encouraging show-ups by giving discounts. We model the restaurant as a service queue and its reservation policy as an advance selling strategy. When customers make reservations in advance, they face uncertainty in the value of (future) consumption. Reservation holders commit to show up and bear the risk of incurring a no-show penalty in return for a no-wait guarantee. We solve for the restaurant's optimal pricing policy and no-show penalty. Our analysis suggests that restaurants should charge a no-show penalty as high as the price of meal while giving a discount to reservation holders for the meal. These results are consistent with the current practice of some high-end restaurants selling non-refundable tickets for their prix-fixe menus where ticket holders lose the face value of the ticket when they fail to show up. Also, there are online reservation systems that give discounts to customers making restaurant reservations through their website. Our results also suggest that as a restaurant faces a larger potential market, it should allocate less capacity for reservation customers. When the market size exceeds a certain threshold, the restaurant is better off if it stops taking reservations.

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1. Introduction

Restaurant reservations can be a double-edged sword. They attract delay-sensitive customers by having a table ready by the time they show up. However, since customers make reservations when they are uncertain about their schedule, they sometimes show up late, cancel at the last moment, or do not show up. The average no-show rate for restaurants in big cities is around 20%, and for an industry with margins as low as 3% to 5%, no-shows can lead to significant losses especially when restaurants are turning away customers waiting outside (Reddy, 2012).

Although customers may think that restaurant industry is all about food, restaurant owners have a consensus that the industry is about space as well. They consider the open space at dining tables to be the same as a manufacturing machine in a factory. When the machine is not producing, they are losing money. To keep the machine running, restaurant owners have started to fight back against no-shows. Some restaurants keep track of customers who have failed to honor their reservations and deny future reservations from these customers. Other restaurants reveal the identities of no-show customers on social media, such as Facebook or Twitter, with the hope of curtailing such behavior. Another common practice is to ask for customers' credit card information when they make reservations so that restaurants can charge a no-show fee when customers do not show up. For example, Le Bernardin, a three Michelin-starred restaurant in New York City with \$127 prix-fixe dinner menus and over \$150 tasting menus, charges \$50 to reservation holders who fail to show up or cancel the reservation 48 hours in advance. Although increasingly many restaurants are adopting the no-show fee policy, customers seem to think that it is unfair for restaurants to charge money for food they have not consumed.

Alternatively, restaurants can encourage reservation holders to show up by offering discounts. An example of this approach is Savored.com, a website through which customers make a reservation for restaurants. The website takes reservations for over 850 restaurants in ten different cities around the U.S. One difference it has from other reservation websites such as OpenTable is that customers receive a discount from 10% to 40% for the meal when they make a reservation through Savored.com. In addition, more and more restaurants are selling discounted gift certificates through Groupon or LivingSocial. The discounts can be redeemed when customers make reservations at the respective restaurants.

With both "stick" and "carrot" approaches in mind, we answer three questions in this paper. First, we study how much no-show penalty restaurants should charge. This is done by solving a restaurant's profit maximization problem over its price and fee. Second, we address whether restaurants can earn more profit by giving discounts to reservation customers. To answer this question, we solve for a restaurant's optimal price discrimination strategy between reservation

customers and walk-in customers. Third, if reservation no-shows are such a big problem, we ask whether restaurants should take any reservation at all. We do this by solving for the optimal capacity allocation between reservation holders and walk-in customers.

We build an analytic model where a restaurant is a monopolistic service queue. When customers walk in to the restaurant without reservations, they incur waiting costs. On the other hand, customers who make reservations receive priority and do not wait; however, they bear the risk of paying a no-show fee if they are unable to keep the reservation. At the time when reservations are made, customers are uncertain about their future valuation of consumption. In this sense, reservations are similar to advance sales in that consumption is temporally separate from purchase. Customers are strategic in that they anticipate future outcomes and make a reservation only if the *ex ante* expected payoff from reservation is higher than that from the outside option of walking in subsequently. In equilibrium, customers make reservation decisions given the restaurant's price and reservation policy, and the restaurant determines its optimal price and no-show fee given customers' behavior.

The remainder of the paper is structured as follows. Section 2 provides literature review. Section 3 describes the model fundamentals. Section 4 provides equilibrium analysis and answers our research questions above. Section 5 discusses potential extensions of the model and Section 6 concludes.

2. Literature Review

This paper is closely related to literature on advance selling. In this literature purchase and consumption are separated and customers make a purchasing decision based on their *ex ante* utility. In Png (1989) a seller with limited capacity sells its capacity to risk averse customers that are uncertain about both the valuation of consumption and the availability of the capacity. The author shows that the most profitable pricing strategy takes the form of reservation that acts as an insurance to customers against stock-out and allows customers to decide to show-up only when their realized valuation is high. Shugan and Xie (2000) show how customers make a purchasing decision when they do not know their valuation of consumption at the time they purchase the product.

DeGraba (1995) and Shugan and Xie (2004) look at the same problem with the consideration of capacity. Shugan and Xie (2005) add competition to the same problem and show how advance selling can increase profitability of the seller under competition. In Gallego and Şahin (2006) customers decide when to purchase a good when their valuations evolve over time. Yu et al. (2008) look at the pricing and capacity decision of a firm in advance selling when customers' valuations are correlated. In Liu and van Ryzin (2008) a firm faces declining prices over time and decides whether to deliberately understock capacity to induce customers to purchase early at higher prices. Aviv and Pazgal (2008) consider the optimal pricing of a finite capacity under contingent pricing strategy and under announced fixed-discount strategy in the presence of forward-looking customers. Fay and Xie (2008) and Jerath et al. (2010) consider a different type of uncertainty in customers' valuation from other papers in advance selling. They study opaque selling where customers make a purchasing decision without knowing exactly what product or service they will receive from a seller that sells multiple distinct items. Nasiry and Popescu (2012) look at how anticipated regret can affect customers' decision and firms' profit in an advance selling context.

There is a stream of research that consider advance selling as means to acquire demand information for a later selling period. Fisher and Raman (1996) look at a quick response system for fashion products where greater portion of production is scheduled in response to initial demand. Tang et al. (2004) suggest an advance booking discount program that attracts customers to commit to their orders prior to selling season. The authors evaluate the benefit of the program and characterize the optimal price when the seller uses the sales information collected from the program to update demand forecast for the selling season. In a similar context, Boyacı and Özalp (2010) study a seller's capacity decision problem with exogenously given prices and Prasad et al. (2011) consider the price and inventory decision when the seller is a newsvendor. In this paper, we apply the advance selling model to the restaurant industry and look at advance selling as means to avoid waiting in a queue rather than ways to secure capacity.

First to introduce the idea of applying revenue management to restaurant industry were Kimes et al. (1998). They address that restaurants use reservations to gain forecast of uncertain arrivals

but do not eliminate the uncertainty because not all customers honor the reservation. Bertsimas and Shioda (2003) studied how many reservations to accept and when to seat walk-in customers using optimization methods. They focused mainly on how to seat the arriving customers and did not model the customers' behavior as a response to the restaurant's strategy as we do in this paper.

The research that studies restaurant reservation systems as a form of advance selling in presence of strategic customers most closely relates to this paper. Alexandrov and Lariviere (2012) look at a restaurant with fixed capacity facing an uncertainty in demand of high or low. Customers *a priori* incur travel cost to visit the restaurant and incur additional cost when the restaurant is full and they cannot receive the service. The authors show that reservation is valuable on nights with small demand because it reduces uncertainty for customers and increases demand, while reservation is costly on busy nights due to no-shows. We look at the same topic (reservation versus walk-in in restaurants) with different focus and setting. We focus on how the restaurant should price its walk-in services and/or reservation services when customers are strategic. The customers who walk-in in our paper do not face uncertainty in finding a seat, but need to wait in the queue before being seated. Çil and Lariviere (2009) assume that advance demand customers who request a reservation are more profitable to the service provider than customers who walk-in. The service provider then decides how much of a limited capacity to allocate to advance customers. The above papers look at capacity decisions, whereas we consider pricing.

Su and Zhang (2009) study a seller's quantity and price decisions when customers face travel cost to shop before knowing the availability of the product they are shopping for. They look at how commitment and availability guarantee can increase the seller's profitability. In our paper customers do not incur travel cost but face disutility of waiting. The uncertainty to customers solely lies on their valuation and not on availability. In this paper we apply the idea of strategic consumers to the restaurant industry where congestion incurs cost to customers.

The last stream of literature our work is related to is the queueing literature that study pricing of the service in a congested server. Naor (1969) studied how a social planner should price a service when customers who arrive to the service center and queue to obtain the service cause

negative externality of congestion to other customers. Customers have homogeneous valuation of the service, incur cost of waiting when standing in the queue, and decide to join the queue or not upon observing the queue length. The queue in our model resembles that of Mendelson and Whang (1990) more in a sense that customers do not observe the exact queue length before deciding to join the queue. Customers make the queueing decision based on their expected waiting cost. Debo and Veeraraghavan (2011) study pricing decision in a queue when price signals the quality of service. Allon et al. (2011) study how informing customers about anticipated service delays can influence customers in service systems. In all of the queueing literature mentioned here, a specific queue type and, subsequently, the formula for waiting time are assumed, but we do not assume any. Yet, we assume that our queue follows general properties of queues studied in the literature. For an extensive review on queueing systems, we refer the readers to Hassin and Haviv (2003).

3. Model

3.1. Model Fundamentals

The Restaurant: The restaurant is a monopoly with a fixed market size equal to Λ and capacity equal to μ . The market size and the capacity are fixed and exogenously given to the restaurant. The restaurant allocates a proportion α of its capacity to serve reservation customers and reserves $1 - \alpha$ of its capacity to accept walk-in customers. In our base model we assume that α is exogenous to the restaurant and the unfilled capacity from reservation no-shows is not reallocated to serve walk-in customers. Later we consider the case where the decision of α is endogenized and the case where the leftover capacity from reservation no-shows is reallocated to walk-in customers. The restaurant's decision is to choose the price, p , and the no-show fee, ϕ , that maximizes its profit. For customers who make a reservation the restaurant prepares a table so that reservation holders do not wait before being served when they show up. Customers walking in without reservations may have to wait in line in order to be served. Thus, to the walk-in customers the restaurant is a queue with an effective arrival rate, $\lambda_w \in [0, \Lambda]$, determined in equilibrium and service rate, $(1 - \alpha)\mu$. Customers do not observe the queue length before walking into the restaurant but know

that their expected waiting time is $w(\lambda_w, (1 - \alpha)\mu)$. We assume that customers do not balk from the queue upon arrival. We do not assume any specific form of queue for the restaurant. However, the expected waiting time function, $w(\lambda, \mu)$, satisfies general convexity conditions: 1) $w(\lambda, \mu)$ is convex and increasing in λ . 2) It is decreasing in μ . 3) Pooling the capacity is beneficial, i.e., $w(n\lambda, n\mu) < w(\lambda, \mu)$ for $n > 1$.

The Customers: Customers are strategic utility maximizers. They are *ex ante* homogeneous but *ex post* differentiated in their valuation of consumption. A customer's valuation, v , consists of two parts, $v = v_0 + \varepsilon$. v_0 is a perception about the restaurant's quality that customers may form based on Zagat rating, number of Michelin stars, or Yelp reviews. Customers form this perception before they make a reservation, and we assume that v_0 is exogenously given, equal for all customers and remains constant. Later in the paper, we relax the assumption of all customers having the same perception of quality, v_0 , and consider a model with heterogeneous customers. Customers face a random shock, ε , in their valuation on the day of consumption. The shock may be due to medical emergency, change of outside option, or simple change of mind. The shock, ε , follows a continuous distribution, G , with density, g , and support, $(-\infty, \infty)$. For notational simplicity, we define a distribution for v to be F with density, f , so that $F(x) = G(x - v_0)$. We use \bar{F} and \bar{G} to denote the complement of F and G , respectively, i.e., $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$. We assume that ε has mean 0 so that the end valuation of consumption, v , is in expectation the same as the initial perception of value, v_0 . Customers' decision is first, whether to make a reservation or to wait based on their expected valuation. On the consumption day, contingent on the realized valuation, reservation holders decide whether to show up or not, and customers without reservations decide whether to walk in or not.

Sequence of Events: Our model is a two period model. In period 1 the restaurant chooses price of meal, p , and no-show penalty, ϕ . Given the price and no-show fee, customers make a reservation without knowing their valuation of consumption. In the beginning of period 2 the shock, ε , realizes,

and customers observe their valuation of consumption, $v = v_0 + \varepsilon$. Reservation holders first decide whether to show up or not. Customers who were not able to secure a reservation in period 1 then decide whether to walk in or stay home. We use the term “advance period” interchangeably with period 1 and “spot period” or “consumption day” interchangeably with period 2. The sequence of events is shown in Figure 1.

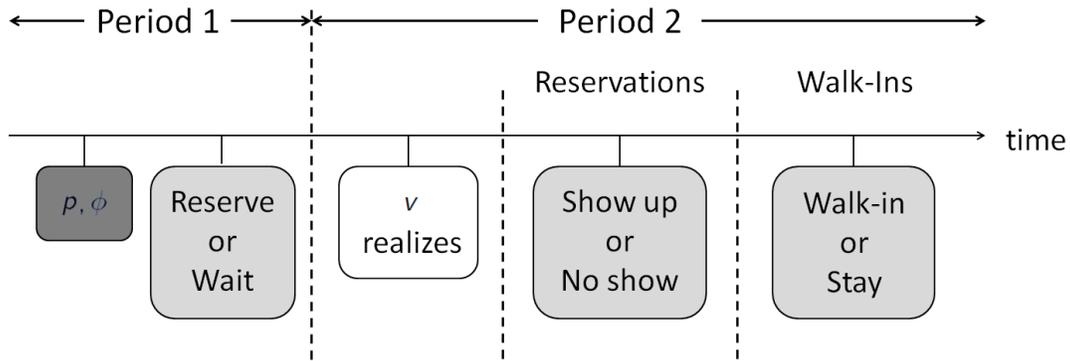


Figure 1 Sequence of Events

3.2. Problem Formulation: Special Cases

In this section we characterize the customers’ equilibrium decisions of making reservations and that of walking in. Then we set up the restaurant’s profit maximization problem given the customers’ equilibrium. For expository purpose, we will consider the special case where the restaurant only offers reservations without accepting walk-ins and the case where the restaurant does not take reservation and accept only walk-ins. Based on the equilibrium we developed in this section, we will analyze results from more general models where the restaurant both takes reservations and accepts walk-ins in the analysis section.

Pure Reservation: Under a pure reservation system the restaurant allocates its entire capacity to serve reservation customers and does not admit any walk-ins, i.e., $\alpha = 1$. The restaurant charges price, p_r , to customers who show up and charges a no-show penalty, ϕ , to customers who do not. On

the consumption day a customer who has made a reservation observes her valuation shock, ε , and decides to show up when the payoff of doing so, $v_0 + \varepsilon - p_r$, is greater than the disutility of paying the no-show penalty, $-\phi$. Recall that reservation holders need not wait before being served when they show up. Therefore, if a customer's realized valuation is $v = v_0 + \varepsilon$, the customer's behavior on the day of service can be summarized as

$$\begin{cases} \text{show up} & \text{if } v \geq p_r - \phi, \\ \text{no show} & \text{if } v < p_r - \phi. \end{cases}$$

Now assuming that all customers are willing to make a reservation beforehand, we can compute the equilibrium show-up rate, λ_r , as the following:

$$\lambda_r(p_r, \phi) = \begin{cases} \Lambda \bar{F}(p_r - \phi) & \text{if } \Lambda < \mu, \\ \mu \bar{F}(p_r - \phi) & \text{if } \Lambda \geq \mu. \end{cases}$$

When market size is less than capacity, the restaurant takes all of the reservations made, while when the market size is greater than the capacity, the restaurant can only take reservations up to its capacity. Multiplying the probability of customers showing up, $\bar{F}(p_r - \phi)$, to the total amount of reservations made gives the rate at which customers show up.

Now, let us consider when customers would indeed make a reservation. Since customers are *ex ante* homogeneous, if making a reservation is attractive, all of the customers will try to make a reservation. When a customer makes a reservation, she does not know her valuation of future consumption, and makes a reservation decision based on her expected utility,

$$E_v[u_r] = E_v \max\{v - p_r, -\phi\} = \int_{p_r - \phi}^{\infty} (v - p_r) f(v) dv - \phi F(p_r - \phi).$$

The customer makes a reservation when her expected gain from making a reservation is greater than the outside option, which we normalize to zero. Therefore, customers make a reservation when $E_v[u_r] \geq 0$. This condition then becomes an individual rationality(IR) constraint or a participation constraint for the restaurant when it sets the price and fee.

Knowing the customers' equilibrium behavior, the restaurant can set its price and fee optimally. Given that customers follow their equilibrium behavior, the restaurant's profit is given by

$$\pi_r(p_r, \phi) = \begin{cases} \Lambda \{p_r \bar{F}(p_r - \phi) + \phi F(p_r - \phi)\} & \text{if } \Lambda < \mu, \\ \mu \{p_r \bar{F}(p_r - \phi) + \phi F(p_r - \phi)\} & \text{if } \Lambda \geq \mu. \end{cases}$$

The restaurant determines p_r and ϕ that maximizes the profit, while satisfying the customers' participation constraint. Therefore, the restaurant's profit maximization problem can be written as

$$\begin{aligned} \max_{p_r, \phi} \quad & p_r \bar{F}(p_r - \phi) + \phi F(p_r - \phi) \\ \text{s.t.} \quad & \int_{p_r - \phi}^{\infty} (v - p_r) f(v) dv - \phi F(p_r - \phi) \geq 0. \end{aligned} \quad (\text{IR}) \quad (1)$$

Pure Walk-in: Under pure walk-in system the restaurant does not take any reservation and reserves its entire capacity to serve walk-in customers, i.e., $\alpha = 0$. The restaurant charges price, p_w , to customers who walk in. In this case customers make a decision of walking in or staying home after realizing their valuation. Customers who walk in face a queue with expected waiting time, $w(\lambda_w, \mu)$, where λ_w is the effective walk-in arrival rate endogenously determined in equilibrium. Therefore, customers who have realized valuation greater than the total cost, $p_w + cw(\lambda_w, \mu)$, walk in, and others stay home. The equilibrium arrival rate, $\lambda_w(p_w)$, as a function of price, p_w , is then determined as

$$\lambda_w(p_w) = \Lambda \bar{F}(p_w + cw(\lambda_w, \mu)).$$

Note that the proportion of people walking in to the restaurant, $\lambda_w(p_w)/\Lambda$, is equal to the probability of customers' valuation being greater than the sum of price and the equilibrium waiting cost, $\bar{F}(p_w + cw(\lambda_w, \mu))$. The marginal customer who walks into the restaurant has valuation, $v = v_0 + \varepsilon$, exactly equal to $p_w + cw(\lambda_w, \mu)$ and is indifferent between walking in and staying home.

Given that the customers behave according to the above equilibrium, the restaurant's profit function can be written as

$$\pi_w(p_w) = p_w \lambda_w(p_w),$$

and the optimal price, p_w^* , is determined as

$$p_w^* = \arg \max_{p_w} p_w \lambda_w(p_w). \quad (2)$$

In order for a unique optimum to exist, the profit function, $p_w \lambda_w(p_w)$, should be unimodal in p_w .

3.3. Problem Formulation: A General Case

We now turn to the setting where a fixed proportion, $\alpha \in (0, 1)$, of total capacity is allocated to serve customers who made reservations and the rest is used to serve customers who walk-in. The capacity set aside for reservation customers who end up not showing up is wasted and is not reallocated to customers who walk-in. We relax this assumption later in the extension. Without reallocation of capacity, the waiting time for walk-in customers is independent of the number of reservation customers that are present at the restaurant.

Under this “hybrid system,” a restaurant charges different price for the meal to walk-in customers from that charged to reservation customers. We call the price for walk-in customers in this hybrid system, p_w , and that for reservation holders who show up on the day of service, p_r . Customers who have made a reservation and fail to show up are charged a no-show fee, ϕ . Again, as customers are *ex ante* homogeneous, all customers will try to make a reservation if the option is attractive in expectation. The restaurant will take the reservation as much as the capacity can accommodate. Then we assume that customers who have been denied a reservation are willing to walk-in on the day of service.

Assume that the price vector, (p_r, ϕ, p_w) , is determined so that making a reservation is attractive to the customers. Then for fixed Λ , μ , and $\alpha \in (0, 1)$, the equilibrium arrival rate, $\lambda_r(p_r, \phi)$ and $\lambda_w(p_w)$, for reservation customers and for walk-in customers, respectively, are given as

$$\lambda_r(p_r, \phi) = \begin{cases} \alpha\mu\bar{F}(p_r - \phi) & \text{if } \Lambda \geq \alpha\mu, \\ \Lambda\bar{F}(p_r - \phi) & \text{if } \Lambda < \alpha\mu, \end{cases}$$

and

$$\lambda_w(p_w) = \begin{cases} (\Lambda - \alpha\mu)\bar{F}(p_w + c\tilde{w}) & \text{if } \Lambda \geq \alpha\mu, \\ 0 & \text{if } \Lambda < \alpha\mu, \end{cases}$$

where $\tilde{w} = w(\lambda_w, (1 - \alpha)\mu)$. Note that as in the pure reservation case, customers holding a reservation show up when their realized valuation net price is greater than the cost of paying the no-show penalty, i.e., $v - p_r > -\phi$. Thus, the proportion of customers showing up is given as $\bar{F}(p_r - \phi)$. Walk-in customers dine out when their realized valuation is greater than the price and the expected waiting cost they need to incur.

Given this equilibrium behavior of customers, the restaurant's profit, π , as a function of prices, (p_r, ϕ, p_w) , is defined as

$$\pi(p_r, \phi, p_w) = \pi_r(p_r, \phi) + \pi_w(p_w),$$

where the reservation profit, π_r , and the walk-in profit, π_w , are respectively given as

$$\pi_r(p_r, \phi) = \alpha\mu \{p_r \bar{F}(p_r - \phi) + \phi F(p_r - \phi)\},$$

and

$$\pi_w(p_w) = (\Lambda - \alpha\mu)p_w \bar{F}(p_w + c\tilde{w}).$$

Now, we look at the condition under which making a reservation is attractive. Customers make a reservation only if making a reservation in expectation has a higher payoff than the outside option of walking in after observing their valuation. This condition can be written as the following inequality, which becomes the incentive compatibility(IC) constraint for the restaurant's profit maximization problem:

$$\int_{p_r - \phi}^{\infty} (v - p_r) f(v) dv - \phi F(p_r - \phi) \geq \int_{p_w + c\tilde{w}}^{\infty} (v - p_w - c\tilde{w}) f(v) dv.$$

Knowing the profit function and the IC constraint, we can write the restaurant's profit maximization problem as

$$\begin{aligned} \max_{p_r, \phi, p_w} \quad & \pi_r(p_r, \phi) + \pi_w(p_w) & (3) \\ \text{s.t.} \quad & \int_{p_r - \phi}^{\infty} (v - p_r) f(v) dv - \phi F(p_r - \phi) - \int_{p_w + c\tilde{w}}^{\infty} (v - p_w - c\tilde{w}) f(v) dv \geq 0. & (IC) \end{aligned}$$

4. Analysis

Recall that we are looking for ways to deal with restaurant reservation no-shows. We consider using two approaches, i.e., "sticks" and "carrots," and our goal is to answer the following three questions: 1) How harsh should the no-show penalty be? 2) Can restaurants earn more profit by giving discounts to reservation holders? 3) Should restaurants take any reservation at all?

4.1. Reservation Policy: How Harsh Should the No-Show Penalty Be?

To answer our first question of what the profit maximizing no-show penalty is we solve for the profit maximization problem given as (3). To focus on the key result, we assume that the price of meal charged to walk-in customers, p_w , is exogenously given as a constant for now. (Endogenizing p_w does not change the result, and we will see how p_w is determined in the next subsection.) Then for any given value of p_w , the profit maximizing meal price charged to reservation customers, $p_r^*(p_w)$, and the optimal no-show fee, $\phi^*(p_w)$, are determined uniquely as follows:

PROPOSITION 1 (Optimal Reservation Price). *For any given $\alpha \in (0, 1)$ and $p_w > 0$, the optimal reservation price $p_r^*(p_w)$, and the optimal no-show fee, $\phi^*(p_w)$, are determined as*

$$p_r^*(p_w) = \phi^*(p_w) = \int_0^\infty v f(v) dv - \int_{p_w + c\tilde{w}}^\infty (v - p_w - c\tilde{w}) f(v) dv. \quad (4)$$

The above proposition shows that the optimal no-show fee should be equal to the price of meal, and this answers our first question. Recall that customers are already feeling unhappy about restaurants charging small amount of no-show penalty. However, our result suggests that for restaurants selling several hundred dollar meals the no-show penalty should be of equivalent level. Although no-show customers do not consume the food, if the restaurant has set aside its capacity for them, they need to pay the full price of the capacity.

This result also provides reasoning for restaurants selling tickets. Under the ticket system, restaurants sell tickets for having a prix-fixe menu on certain date and time. Customers who have bought a ticket show up on time, show the ticket, eat the meal, and leave. If a customer does not show up, then no refund is given, and the price paid for the ticket is wasted. Although framed differently, purchasing a non-refundable ticket and committing to pay the no-show penalty equal to the price of meal have equivalent result in customers' utility.

Note that the optimal reservation price and the no-show fee, given any value of p_w , are the expected valuation of reservation customers that show up, subtracted by the expected utility from the outside option of walking in. The equilibrium outcome is efficient in the sense that customers

with non-negative *ex post* valuation show up. Since customers who do not show up also pay the same “price” as those who show up, the profit of the restaurant does not depend on the proportion of customers showing up.

The optimal pricing policy, (p_r^*, ϕ^*, p_w^*) , can now be obtained by solving the following optimization problem:

$$\max_{p_w} \pi_r(p_r^*(p_w), \phi^*(p_w)) + \pi_w(p_w), \quad (5)$$

where $p_r^*(p_w)$ and $\phi^*(p_w)$ are given as (4).

4.2. Price Discrimination: Should Restaurants Give Discounts to Reservation Customers?

In order to answer our second question, we compare the optimal walk-in price, p_w^* , computed by solving (5), and the optimal reservation price, $p_r^*(p_w^*)$. For a proper comparison, we assume that the market size is large enough, $\Lambda > \mu$, so that there are enough number of customers to fill both the reservation capacity and walk-in capacity. Note that only when a restaurant allocates non-zero capacity to both walk-in customers and reservation customers, it makes sense to price discriminate. Thus, in order to compare the two prices, we assume that the restaurant we consider is optimally allocating positive capacity to serving both walk-in customers and reservation customers, i.e., $\alpha^* \in (0, 1)$. Here, α^* is the optimal capacity allocated for reservation customers, determined by solving the following optimization problem:

$$\begin{aligned} \max_{\alpha, p_r, \phi, p_w} \quad & \pi_r(p_r, \phi) + \pi_w(p_w) \\ \text{s.t.} \quad & \int_{p_r - \phi}^{\infty} (v - p_r) f(v) dv - \phi F(p_r - \phi) - \int_{p_w + c\tilde{w}}^{\infty} (v - p_w - c\tilde{w}) f(v) dv \geq 0 \quad \text{if } \alpha > 0, \\ & \alpha \in [0, 1]. \end{aligned} \quad (6)$$

Note that now α is endogenized as another layer of restaurant’s decision, whereas in the previous section the profit maximization problem, (3), assumed α as an exogenously given parameter.

When the restaurant optimally chooses to serve both reservation customers and walk-in customers, we show that the optimal price for reservation customers is strictly less than the optimal price for walk-in customers. The following proposition formally states this.

PROPOSITION 2 (**Price Discrimination**). *Assume that $\Lambda > \mu$ and the optimal allocation, α^* , satisfies $\alpha^* \in (0,1)$. Then,*

$$p_r^* < p_w^*.$$

Hereby, we answer our second question of whether restaurants can be better off by giving discounts to reservation holders who show up, and the answer is yes. This means that if a restaurant is to price discriminate, it needs to give a discount to customers who pay money before realizing their valuation of consumption and bear the risk of paying a no-show penalty. Because walking in is an outside option to making a reservation, there is a pressure for the restaurant to make the inside option more attractive by charging a lower price. Note also that the equilibrium walk-in demand is always strictly less than the capacity set aside for them due to wait cost in queue, i.e., $\lambda_w < (1 - \alpha)\mu$, while demand under reservation system is perfectly matched with supply. Then, along with the no-show fee equal to the price, capacity set aside for reservation earns profit to the full extent, whereas capacity allocated for walk-ins is not fully utilized. Therefore, if the optimal price for reservation were higher, it would be more profitable for the restaurant to set its entire capacity to take reservations, thus, contradicting our assumption that the restaurant is optimally allocating positive capacity to walk-in customers.

4.3. Capacity Allocation: Should Restaurants Take Reservations?

Now, we answer our last question: If reservation no-shows are such a plaguing problem, should the restaurant take any reservations at all? To answer this question, we solve for the optimal capacity allocation problem, (6), and let $\alpha^*(\Lambda)$ denote the optimal proportion of capacity allocated to reservation customers as a function of market size, Λ , with the capacity fixed to μ .

We begin our analysis by comparing the restaurant's profit under two special case models, pure reservation ($\alpha = 1$) and pure walk-in ($\alpha = 0$), when the market size and the capacity are equal.

Under pure reservation system, solving (1) gives the following result:

LEMMA 1 (**Equilibrium for Pure Reservation**). *For fixed Λ , μ , and F , the optimal price vector, (p_r^*, ϕ^*) , under pure reservation system is given as*

$$p_r^* = \phi^* = \int_0^\infty v f(v) dv.$$

The equilibrium show-up rate is

$$\lambda_r(p_r^*, \phi^*) = \begin{cases} \Lambda \bar{F}(0) & \text{if } \Lambda < \mu, \\ \mu \bar{F}(0) & \text{if } \Lambda \geq \mu, \end{cases}$$

and the optimal profit for the restaurant is

$$\pi_r(p_r^*, \phi^*) = \begin{cases} \Lambda p_r^* & \text{if } \Lambda < \mu, \\ \mu p_r^* & \text{if } \Lambda \geq \mu. \end{cases}$$

On the other hand, the optimal profit and price for pure walk-in system is given by solving (2).

Now, we compare the two optimal profits and address that a restaurant with small market size is better off under pure reservation system, whereas a restaurant with large potential market performs better under pure walk-in system.

PROPOSITION 3 (**Market Size**). *Let $\pi_w^*(\Lambda, \mu)$ and $\pi_r^*(\Lambda, \mu)$ be the optimal profit for the restaurant as a function of parameters, Λ and μ , under pure walk-in system and that under pure reservation system, respectively. Then given a fixed μ and the valuation distribution, F , there exists $\tilde{\Lambda}$ such that*

$$(i) \pi_w^*(\Lambda, \mu) \leq \pi_r^*(\Lambda, \mu), \text{ for all } \Lambda \leq \tilde{\Lambda},$$

$$(ii) \pi_w^*(\Lambda, \mu) > \pi_r^*(\Lambda, \mu), \text{ for all } \Lambda > \tilde{\Lambda}.$$

To see why pure reservation system outperforms when the potential market size is small, assume that the market size is smaller than the capacity. In this case, the restaurant can take all reservations that are made. Then assume that under pure reservation system the restaurant charges $(p_r, \phi) = (p, 0)$, and under pure walk-in system it charges $p_w = p$. Then since customers who make a reservation do not have to wait, more customers will show up under reservation system than under walk-in system so that reservation system is more profitable in this case. When the restaurant

optimizes for the price and the no-show penalty for the reservation holders it will perform even better.

Now to see why the opposite is true with large market size let us fix λ_w and increase the potential market size, Λ . Then the proportion of customers who walk in, λ_w/Λ , decreases so that the valuation of marginal customer that walks in, $p_w + cw(\lambda_w, \mu)$, increases. Since λ_w is fixed, the waiting cost is also kept constant so that p_w should be increased. In other words, when the market size becomes larger, a pure walk-in restaurant can increase the price while maintaining the same demand. However, the optimal price and fee for reservation system is independent of the market size and solely dependent on the distribution of customers' valuation, $p_r^* = \phi^* = \int_0^\infty v f(v) dv$. Moreover, even when the demand is very large, a restaurant can take reservation only up to its capacity under pure reservation system. Thus, the profit for reservation system is strictly bounded above regardless of the market size. Therefore, when market size becomes larger, the profit under pure walk-in system catches up with that under pure reservation system and remains to be higher for any larger market size.

Next, we consider how scaling up the demand and the capacity changes a restaurant's optimal profit under walk-in system and that under reservation system.

PROPOSITION 4 (Economies of Scale). *Let $\pi_w^*(\Lambda, \mu)$ and $\pi_r^*(\Lambda, \mu)$ be the optimal profit for the restaurant as a function of parameters, Λ and μ , under pure walk-in system and that pure reservation system, respectively. For any given Λ , μ , and $n > 1$,*

$$(1) \pi_w^*(n\Lambda, n\mu) > n\pi_w^*(\Lambda, \mu), \text{ (increasing returns to scale)}$$

$$(2) \pi_r^*(n\Lambda, n\mu) = n\pi_r^*(\Lambda, \mu). \text{ (constant returns to scale)}$$

We find that scaling the market size and capacity by the same constant, $n > 1$, scales the walk-in profit by greater than n , while the same scaling in demand and capacity only scales the reservation profit exactly by n . This means that pure walk-in system has increasing returns to scale, while pure reservation system has constant returns to scale. This may partially explain why McDonald's expanded the number of franchise stores to serve the large demand, whereas Bibou, a Philadelphia

based reservation-only French restaurant whose reservation slots run out a month in advance for its popularity, sticks to its narrow place and small number of staffs. McDonald's may benefit from increasing returns to scale when it expands its capacity according to its popularity, while Bibou may not have that advantage of increasing returns to scale in expanding.

With the intuition developed above, we now discuss the result of the capacity allocation problem, (6), which is summarized in the following proposition:

PROPOSITION 5 (Optimal α). *Let $\Lambda' > \Lambda > \mu$. Then, corresponding optimal capacity allocation, $\alpha^*(\Lambda')$ and $\alpha^*(\Lambda)$ satisfy:*

$$\alpha^*(\Lambda') \leq \alpha^*(\Lambda),$$

and

$$\lim_{\Lambda \rightarrow \infty} \alpha^*(\Lambda) = 0.$$

This result conforms to our previous finding in Proposition 3. In Proposition 3 we stated that for a restaurant with greater market size, pure walk-in system brings higher profit than pure reservation system. This is also true in the hybrid system, so that optimal capacity allocated to reservation decreases as the market size increases. Moreover, when the market size increases above a certain value, it shows that operating as a pure walk-in restaurant performs better than accepting any reservations. Note that this is similar to setting a protection limit in a revenue management problem. Our result suggests that when potential market size is large, restaurants should set the protection limit to its entire capacity.

4.4. Numerical Results

To fathom how much more profit a restaurant can earn by adopting the stick and carrot approaches we have talked about, we run numerical studies with carefully selected parameter values. Recall that the parameter values our model require are the potential market size, Λ , capacity, μ , distribution of valuation, F , and the per-unit waiting cost, c . We picked a restaurant in Philadelphia, named Buddakan, and borrowed model parameters from the restaurant. Buddakan is a large restaurant

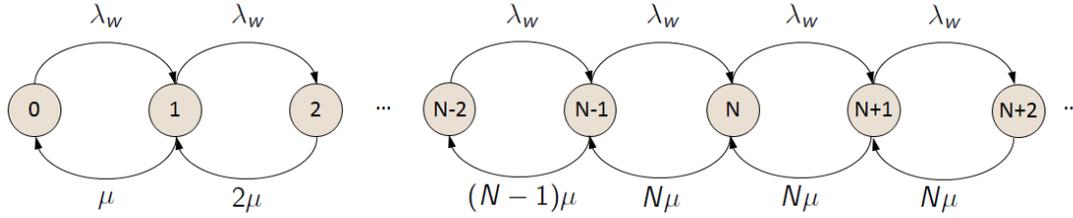


Figure 2 Markov Chain

with 175 seats in total but has tables they use to serve special event parties. Excluding those special purpose seats gives us about 100 seats with 40 tables, which gives average table size of 2.5 people. Meal duration is about 2 hours, and the average bill size is \$50/person, i.e., \$125/table.

Although we do not assume any specific form of queue in our model, depending on the characteristics of a restaurant's operation, we can impose a particular queueing model. The bottleneck of Buddakan's operation is the table turnover, so we model a server of our queue to be a table rather than a wait person or a kitchen with $N = 40$ servers in the queue. Then, since the meal duration is 2 hours, the service rate, μ , can be computed to be 0.5 tables/hour. With the information given so far, we can construct a Markov chain that represents the queue with state space, $\mathcal{S} = \{0, 1, 2, \dots\}$, being the number of customers as in Figure 2. When the state is $s \geq N$, then all of the tables at the restaurant are occupied, and newly arriving customers need to wait before being seated. The transition rate from state, $s \geq N$, to $s - 1$ is equal to $N\mu$, which is the total number of servers multiplied by the service rate per server. From state, $s < N$, to $s - 1$ is given as $s\mu$, the number of tables occupied multiplied by the per-server service rate. Transition rate from any state s to $s + 1$ is given as the equilibrium walk-in arrival rate, λ_w . Here, the arrival rate is determined as number of tables arriving to the restaurant per unit time. Note that the Markov chain then happens to coincide with $M/M/N$ queue. By setting up the balance equation for each state, one can come up with the steady state probability and consequently the expected number of customers in the restaurant. Then, by Little's law, we can compute the expected waiting time for the walk-in customers.

We choose the cost of waiting, c , as \$18.125/hour ($=\$7.25/\text{hour} \times 2.5$ people/table) for each

table since \$7.25 is the federal minimum hourly wage. (However, our numerical result is robust to small changes in the value of c .) Buddakan both takes reservations and accepts walk-ins and fills 60% of its capacity with reservation customers, hence, $\alpha = 0.6$. To infer the distribution of customers' valuation, v , we use the fact that the average no-show rate in Philadelphia is 20%. We assume that customers enjoy a surplus of 25% of the price they pay from their consumption. For example, if a table of customers pay \$125 for a meal, they enjoy the valuation of \$150 on average. Our results remain in similar orders of magnitude for consumer surplus of up to 200%. Currently, Buddakan does not charge any no-show fee, so the reservation holders do not show up when their realized valuation is less than the price they have to pay, \$125. Given that the cutoff value is \$125, we can infer the distribution of valuation as $N(\$150, \$30)$ for a table.

Now, we compute the optimal profit over the ratio between market size and capacity under three different options: (i) charging a no-show fee and giving a discount to reservation customers, i.e., $\phi = p_r < p_w$, (ii) charging a no-show penalty but keeping the price equal between reservation holders and walk-in customers, i.e., $\phi = p_r = p_w$, (iii) neither charging a no-show fee nor giving a discount to reservation customers, where $\phi = 0$ and $p_r = p_w$. The results are summarized in Figure 3.

As shown in Figure 3, on less busy nights, the restaurant benefits greatly by giving discounts to customers that make a reservation and show up. For example, when the potential market size is 1.5 times the total capacity, $N\mu$, then the model suggests Buddakan that it should charge \$123 for the meal sold to reservation customers and \$128 for that sold to walk-in customers. The restaurant enjoys 7.4% profit increase by giving that 5% discount to reservation customers. Charging no-show penalty equal to the price of meal for reservation customers without giving discount does not bring much profit increase when the potential market size is small. On the other hand, on busy nights, the benefit from no-show fee dominates that from price discrimination. When the potential market size is twice the total capacity, the restaurant can earn 14.5% more profit by charging no-show fee equal to the price of meal, whereas giving discounts to reservation customers bring relatively less profit increase of 2.5%. Therefore, Buddakan can consider giving discounts to reservation customers on slower time slots such as Monday 5pm and charging no-show penalty on busy weekends.

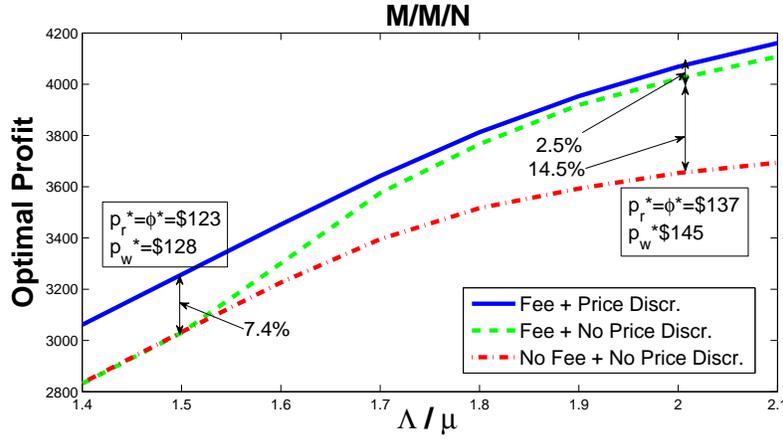


Figure 3 Optimal Profit over Λ/μ

5. Extensions and Discussion

5.1. Heterogeneous Customers

In this section, we relax the assumption of all customers having equal *ex ante* valuation and analyze the restaurant's optimal pricing strategy when the customers have different perception of valuation, v_0 . We assume that a customer is either of two types, high or low. We denote $i \in \{H, L\}$ to be a superscript that represents the type of a customer. Customers' *ex ante* perception of valuation is given as

$$v_0^i = \begin{cases} v_0^H, & \text{with probability } q, \\ v_0^L, & \text{with probability } 1 - q, \end{cases}$$

where $v_0^H > v_0^L$. $q \in [0, 1]$ is the proportion of high-type customers among the potential market, which is exogenously given and fixed. We call a customer to be of high-type if her valuation perception is $v_0^i = v_0^H$, and of low-type if $v_0^i = v_0^L$. Then, type- i customer's *ex post* valuation of consumption is given as $v^i = v_0^i + \varepsilon$ with ε being her valuation shock, which realizes on the consumption day. Given the restaurant's price vector, (p_r, ϕ, p_w) , and the capacity allocation for reservation, α , type- i customers' expected utility of making a reservation and that of walking-in on the spot period are given as

$$E_\varepsilon[u_r^i] = E_\varepsilon \max\{v_0^i + \varepsilon - p_r, -\phi\} = \int_{p_r - \phi - v_0^i}^{\infty} (v_0^i + \varepsilon - p_r) g(\varepsilon) d\varepsilon - \phi G(p_r - \phi - v_0^i),$$

and

$$E_\varepsilon[u_w^i] = E_\varepsilon \max\{v_0^i + \varepsilon - p_w - cw(\lambda_w, (1 - \alpha)\mu), 0\} = \int_{p_w + c\tilde{w} - v_0^i}^{\infty} (v_0^i + \varepsilon - p_w - c\tilde{w})g(\varepsilon)d\varepsilon,$$

respectively, where \tilde{w} is the equilibrium expected waiting time for walk-in customers. Type- i customers then make a reservation when $E_\varepsilon[u_r^i] \geq E_\varepsilon[u_w^i]$.

With heterogeneous customer types, there are two possible strategies for the restaurant: (i) Charge the reservation price and the no-show fee to the level that only high-type customers make reservations. (ii) Charge the price and the fee so that both high-type and low-type customers make reservations. Note that high-type customers gain higher utility from making a reservations than low-type customers, and the case where only low-type customers making reservations is infeasible. The incentive compatibility constraint for the restaurant then becomes

$$E_\varepsilon[u_r^H] \geq E_\varepsilon[u_w^H], \quad \text{if strategy (i) is adopted,}$$

$$E_\varepsilon[u_r^L] \geq E_\varepsilon[u_w^L], \quad \text{if strategy (ii) is adopted.}$$

For each strategy, the restaurant solves its profit maximization problem given as (3) subject to corresponding IC constraint. After comparing the maximum profits that can be earned from the two strategies, the restaurant decides its price vector, (p_r, ϕ, p_w) . By solving this profit maximization problem, we can show the following:

PROPOSITION 6. *Let (p_r^*, ϕ^*, p_w^*) be the optimal price vector for the restaurant. Then given Λ and μ , there exists $\tilde{q} \in (0, 1)$, such that*

(i) for $q \geq \tilde{q}$, only high-type customers make a reservation, and

$$\phi^* = p_r^* = \int_{-v_0^H}^{\infty} (v_0^H + \varepsilon)g(\varepsilon)d\varepsilon - \int_{p_w^* + c\tilde{w} - v_0^H}^{\infty} (v_0^H + \varepsilon - p_w^* - c\tilde{w})g(\varepsilon)d\varepsilon,$$

(ii) for $q < \tilde{q}$, both high-type and low-type customers make a reservation, and

$$\phi^* < p_r^*.$$

If the optimal capacity allocated to reservation customers, α^ , satisfies $\alpha^* \in (0, 1)$, then for any value of $q \in (0, 1)$,*

$$p_r^* < p_w^*.$$

The above proposition shows that when there are enough proportion of customers that have high perceived valuation, the restaurant should have only the high-type customers make reservations. In this case, the restaurant charges the no-show penalty equal to the price of meal. However, as more customers perceive the quality of the restaurant to be low, the restaurant can no longer charge the no-show penalty as high as the price. This may explain why we observe only a few restaurants with multiple Michelin stars charge the no-show penalties as high as the price, while most other restaurants with average quality charge lower fees for no-shows if at all. For example, Chef's Table at Brooklyn Fare in New York City, which has three Michelin stars and sells \$225 prix-fixe tasting menu, is one of very few restaurants that charge a no-show penalty equal to the price. Also, recall that the restaurants that sell tickets for their tasting menus (e.g., Next, Alinea, and Saison) are at least two Michelin-starred. On the other hand, giving discounts to reservation customers remains to be optimal under the existence of heterogeneous customer valuation. This is probably why we see a wide range of restaurants taking reservations through Savored.com.

5.2. Reallocating Unclaimed Capacity in Hybrid System

In the preceding analyses we did not allow the restaurant to reallocate the unclaimed seats from reservation no-shows to customers who are walking in. Now, we relax the assumption and see how the optimal price and no-show fee change. For simplicity, we assume that the restaurant charges the same price to both reservation customers and walk-in customers. Note that when the market size is smaller than the capacity, all customers can secure a reservation in the advance period so that no customers are left to walk-in in the spot period. Therefore, we focus on the case where $\Lambda > \mu$ in this section.

Let (p^r, ϕ^r) denote the price and the no-show fee charged by the restaurant, and λ_r^r and λ_w^r denote the equilibrium arrival rate from reservation customers and walk-in customers, respectively. When the restaurant reallocates unclaimed capacity from reservation no-shows to customers that walk in, the expected waiting time for walk-in customers in equilibrium is given as

$$\hat{w} = w(\lambda_w^r, (1 - \alpha)\mu + \alpha\mu F(p^r - \phi^r)),$$

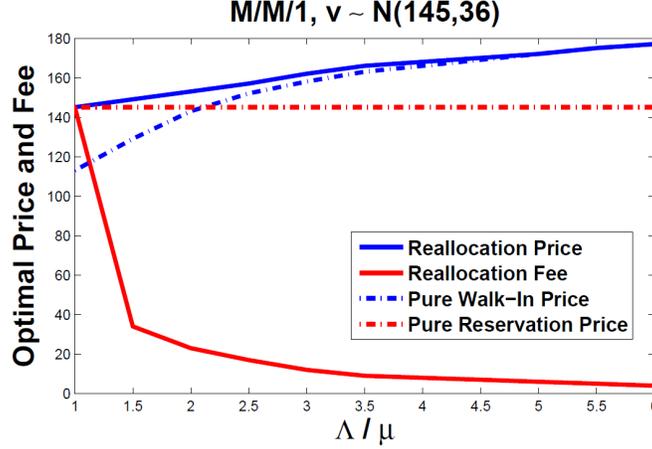


Figure 4 Reallocation Price and Fee

and

$$\begin{cases} \lambda_r^r = \alpha\mu\bar{F}(p^r - \phi^r), \\ \lambda_w^r = (\Lambda - \alpha\mu)\bar{F}(p^r + c\hat{w}). \end{cases}$$

Note that walk-in customers are now served not only by the capacity initially allocated to walk-in customers, $(1 - \alpha)\mu$, but also by the unclaimed capacity from reservation no-shows, $\alpha\mu\bar{F}(p^r - \phi^r)$. Then the restaurant can determine its optimal price and fee by solving the following profit maximization problem.

$$\begin{aligned} \max_{p^r, \phi^r} \quad & p^r \lambda_r^r + \phi^r (\mu - \lambda_r^r) + p^r \lambda_w^r \\ \text{s.t.} \quad & E_v[u_r^r] \geq E_v[u_w^r]. \end{aligned} \quad (\text{IR})$$

In this section, we borrow model parameters from a restaurant in Philadelphia named Union Trust to numerically compute the optimal price and no-show fee under the existence of capacity reallocation. The restaurant is a fine dining steak house with average bill size of \$115/person including tax and tips. It has 170 seats, and the average meal duration is about 2 hours so that the service rate is given as 85 customers/hr. For Union Trust we use $M/M/1$ queue to model the kitchen that serves a large number of customers. This high-end steak house runs mostly on reservation customers, and α is approximately 0.8. As explained in the Numerical Study section, we infer the distribution of customer valuation to be $N(\$145, \$36)$ based on a no-show rate of 20%. Again, we assume $c = \$7.25$ per person per hour.

The optimal price vector, (p^{r*}, ϕ^{r*}) , and the optimal profit over Λ/μ are shown in Figure 4. Note that now with reallocation of unclaimed capacity, the optimal no-show penalty, ϕ^{r*} , is strictly less than the optimal price of meal, p^{r*} ,

$$\phi^{r*} < p^{r*},$$

and the optimal no-show fee, ϕ^{r*} , decreases to zero as the ratio goes up.

In reality, a restaurant usually keeps a reserved table for 15 minutes. It calls the reservation holder when the customer does not show up by then and gives away the table to a customer waiting in line when it thinks the reservation holder will not show up. This may be a reason for not many restaurants charging such high no-show penalty in reality.

5.3. Overbooking

In this section we consider the option to overbook under pure reservation system. Note that overbooking affects the restaurant's operation only when the market size is greater than the capacity, and thus, we will focus on the case, where $\Lambda > \mu$, for this section. When the market size is smaller than or equal to the capacity, the analysis is equivalent to that under pure reservation system without overbooking. As in the case without overbooking, the restaurant charges price, p_r^o , to customers who make a reservation and show up on the day of service and charges a fee, ϕ^o , for those who do not show up after making a reservation. Given p_r^o and ϕ^o , the equilibrium behavior of customers duplicates that under pure reservation system without overbooking.

Given the equilibrium behavior of customers, the restaurant anticipates that fraction $F(p_r^o - \phi^o)$ of customers who have made a reservation will not show up on the day of service. Thus, the restaurant can, in expectation, accept up to $\mu/\bar{F}(p_r^o - \phi^o)$ reservations without having to reject any reservation-holding customers due to shortage of seats on the day of service. The restaurant's profit under overbooking is then given as $\pi_r^o(p_r^o, \phi^o) = \min\{\Lambda, \mu/\bar{F}(p_r^o - \phi^o)\} \{p_r^o \bar{F}(p_r^o - \phi^o) + \phi^o F(p_r^o - \phi^o)\}$. This expression shows that when the potential market size Λ is smaller than the amount of reservation the restaurant can handle, the restaurant takes Λ reservations. The restaurant's profit maximization problem is then given as

$$\begin{aligned} \max_{p_r^o, \phi^o} \quad & \min \{ \Lambda, \mu / \bar{F}(p_r^o - \phi^o) \} \{ p_r^o \bar{F}(p_r^o - \phi^o) + \phi^o F(p_r^o - \phi^o) \} \\ \text{s.t.} \quad & \int_{p_r^o - \phi^o}^{\infty} (v - p_r^o) f(v) dv - \phi^o \bar{F}(p_r^o - \phi^o) \geq 0. \end{aligned}$$

Solving the above optimization problem gives the optimal price policy, (p_r^{o*}, ϕ^{o*}) , as the following proposition suggests.

PROPOSITION 7 (Optimal Overbooking Price and Fee). *For fixed Λ and μ , with $\Lambda > \mu$, the optimal price vector, (p_r^{o*}, ϕ^{o*}) , is given as the following:*

(i) *If $\Lambda \leq \mu / \bar{F}(0)$:*

$$p_r^{o*} = \phi^{o*} = \int_0^{\infty} v f(v) dv.$$

(ii) *If $\Lambda > \mu / \bar{F}(0)$:*

$$\int_{p_r^{o*} - \phi^{o*}}^{\infty} v f(v) dv - p_r^{o*} \bar{F}(p_r^{o*} - \phi^{o*}) - \phi^{o*} F(p_r^{o*} - \phi^{o*}) = 0,$$

$$\mu = \Lambda \bar{F}(p_r^{o*} - \phi^{o*}),$$

$$p_r^{o*} > \phi^{o*}, \quad p_r^{o*} > p_r^*, \quad \phi^{o*} < \phi^*.$$

The result states that when the market size is relatively small, our previous result under pure reservation system without overbooking where the no-show fee being equal to the price of meal remains to hold. Also, the price and the fee with overbooking equal to those under pure reservation system without overbooking. In this case, the restaurant takes reservations up to Λ , and $\Lambda \bar{F}(0) \leq \mu$ of the customers will show up in expectation. On the other hand, when the market size becomes sufficiently large, i.e., if $\Lambda > \mu / \bar{F}(0)$, then it is optimal for the restaurant to charge a higher price and a lower fee. The restaurant takes all of the reservations it receives, Λ , and sets the price and fee so that just the right number of customers the restaurant can handle will show up. Therefore, the price and fee satisfies $\mu = \Lambda \bar{F}(p_r^{o*} - \phi^{o*})$, and as a result, the price, p_r^{o*} , is greater than ϕ^{o*} . This suggests that the restaurant may appear to set a higher price and lower no-show fee so that it can profit from no-show fees even when some customers do not show up. Also, since the market size is large and the restaurant takes the reservation from the entire market, it is able to charge its price high without wasting its capacity.

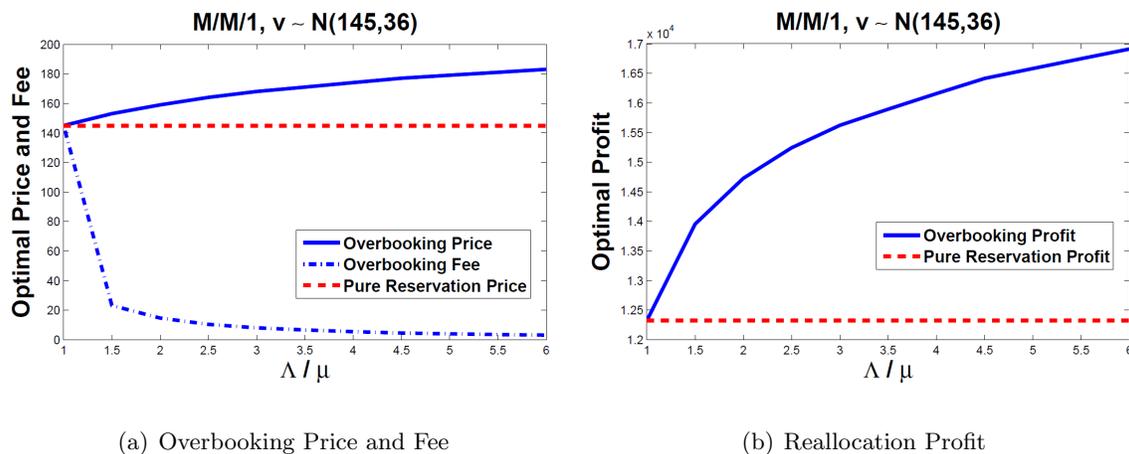


Figure 5 Reallocation Price and Profit over Λ/μ

Again, we use the Union Trust example to visualize the above result and show how profitable overbooking policy is. We assume that the queue is of $M/M/1$ type and keep all other parameter values the same as before. First, we show in Figure 5(a) that as the market size grows larger with respect to the capacity, the optimal price with overbooking increases, whereas the corresponding no-show fee decreases. Also, Figure 5(b) shows that the additional profit earned by overbooking increases as the ratio between market size and capacity increases. In our example, as Λ increases from μ to 6μ , the average profit increased is 28.4% on average.

Note that the above result showing overbooking as such an attractive policy is when the profits were compared *ex ante*. While the advantage of overbooking seems outstanding, there is always a chance that everybody will show up *ex post*. Although this is a rarity for large restaurants that are protected by the law of large numbers, smaller restaurants should exercise caution when adopting overbooking policies.

6. Conclusion

Reservation no-shows lead to wasted capacity in restaurants. In this paper we consider two remedies: to punish no-shows by charging fees and to encourage show-ups by giving discounts. We build an analytic model where a restaurant is represented as a service queue and the reservation policy is modeled as an advance selling strategy, where customers make the commitment to show up when they are uncertain about the valuation of consumption in return for a no-wait guarantee. We

solve for the optimal price and no-show penalty, and our results offer recommendations on what restaurants should do.

We make three main suggestions to restaurants. First, we suggest that restaurants should charge no-show customers a penalty equal to the price they charge to customers who show up. Although charging such high no-show fee seems surprising and rare in practice, there are some high-end restaurants such as Chef's Table in Brooklyn Fare that currently charge the no-show penalty equal to the price of their meals. By considering *ex ante* heterogeneous customers, we explain why lower-end restaurants may not charge such high penalty. Second, we advise that incentives should be given to reservation customers to encourage them to show up. In other words, it is optimal for the restaurant to give discounts to customers who make reservations when they are yet uncertain about their consumption valuation. In numerical examples, we find that a combination of the no-show penalty and price discrimination strategies brings about a 20% profit increase for a realistic range of parameter values. Third, by solving for the optimal capacity allocation between reservations customers and walk-in customers we show that it is optimal for restaurants to allocate less and less capacity to take reservations as the market size increases. In particular, when the market size becomes greater than a certain threshold, it is optimal for the restaurants to operate as a pure walk-in restaurant.

The recommendations above have solid theoretical support, but there are additional considerations when it comes to implementing them in practice. How would the customers feel about paying such a high no-show penalty and being charged a different price from other customers? To address these issues there are multiple directions to take. One way is to model the interaction between the restaurant and customers as a repeated game, where restaurants have to consider future customer goodwill. Another broad topic is to incorporate hidden information and reputation concerns. For example, when restaurant quality is unknown, the reservation policy can be a useful signaling device. Finally, we can incorporate behavioral concerns such as loss aversion and fairness. Even in the simplest one-shot settings, restaurant patrons may not respond to price discrimination and

no-show penalties in the perfectly rational manner as modeled in this paper. Future work can explore how to fine-tune our recommendations.

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Appendix

Proof of Statements

Proof of Proposition 1 Let $u_w(p_w)$ and $u_r(p_r, \phi)$ denote the *ex ante* utility of a customer from reservation and walk-in, respectively, under hybrid system when the price charged from the

restaurant is (p_w, p_r, ϕ) . Then

$$u_w(p_w) = \int_{p_w+c\tilde{w}}^{\infty} (v - p_w - c\tilde{w})f(v)dv,$$

$$u_r(p_r, \phi) = \int_{p_r-\phi}^{\infty} (v - p_r)f(v)dv - \phi F(p_r - \phi),$$

and the expected profit for the restaurant under price policy, (p_w, p_r, ϕ) , satisfies

$$\pi(p_w, p_r, \phi) = \pi_w(p_w) + \pi_r(p_r, \phi),$$

where the walk-in profit, π_w , and the reservation profit, π_r , are respectively given as

$$\pi_w(p_w) = (\Lambda - \alpha\mu)p_w\bar{F}(p_w + c\tilde{w})$$

$$\pi_r(p_r, \phi) = \alpha\mu \{p_r\bar{F}(p_r - \phi) + \phi F(p_r - \phi)\}.$$

Then the following four statements are true.

- (i) If p_r is fixed, $u_r(p_r, \phi)$, decreases in ϕ : $\because du_r(p_r, \phi)/d\phi = -F(p_r - \phi) < 0$.
- (ii) If ϕ is fixed, $u_r(p_r, \phi)$, decreases in p_r : $\because du_r(p_r, \phi)/dp_r = -\bar{F}(p_r - \phi) < 0$.
- (iii) If $p_r < \phi$, $\pi_r(p_r, \phi)$ increases in p_r : $\because d\pi_r(p_r, \phi)/dp_r = \bar{F}(p_r - \phi) + (\phi - p_r)f(p_r - \phi) > 0$.
- (iv) If $p_r > \phi$, $\pi_r(p_r, \phi)$ is increases in ϕ : $\because d\pi_r(p_r, \phi)/d\phi = F(p_r - \phi) + (p_r - \phi)f(p_r - \phi) > 0$.

Recall that the restaurant's profit maximization problem is written as

$$\begin{aligned} & \max_{p_w, p_r, \phi} \pi_w(p_w) + \pi_r(p_r, \phi) \\ & \text{s.t.} \quad \int_{p_r-\phi}^{\infty} (v - p_r)f(v)dv - \phi F(p_r - \phi) - \int_{p_w+c\tilde{w}}^{\infty} (v - p_w - \tilde{w})f(v)dv \geq 0. \end{aligned} \quad (IR)$$

Claim1: The optimal price vector, (p_w^*, p_r^*, ϕ^*) , satisfies

$$\int_{p_r^*-\phi^*}^{\infty} (v - p_r^*)f(v)dv - \phi^* F(p_r^* - \phi^*) = \int_{p_w^*+c\tilde{w}}^{\infty} (v - p_w^* - \tilde{w})f(v)dv.$$

\because When $\int_{p_r-\phi}^{\infty} (v - p_r)f(v)dv - \phi F(p_r - \phi) - \int_{p_w+c\tilde{w}}^{\infty} (v - p_w - \tilde{w})f(v)dv > 0$, by statement (iii) and (iv), the restaurant can either increase p_r or ϕ and increase $\pi_r(p_r, \phi)$. Since $u_r(p_r, \phi)$ decreases in p_r and ϕ , the optimal (p_r^*, ϕ^*) satisfies $\int_{p_r^*-\phi^*}^{\infty} (v - p_r^*)f(v)dv - \phi^* F(p_r^* - \phi^*) - \int_{p_w^*+c\tilde{w}}^{\infty} (v - p_w^* - \tilde{w})f(v)dv = 0$, i.e., the participation constraint is binding in the optimum.

Claim2: For a fixed p_w , setting (p_r, ϕ) to be $p_r > \phi$ cannot be optimal for the restaurant.

\therefore Suppose (p_r, ϕ) and (p'_r, ϕ') satisfy $p_r > \phi$, $p'_r = p_r - \epsilon$,

$$u_r(p_r, \phi) = \int_{p_r - \phi}^{\infty} (v - p_r) f(v) dv - \phi F(p_r - \phi) = \int_{p_w + c\tilde{w}}^{\infty} (v - p_w - c\tilde{w}) f(v) dv, \quad (7)$$

$$u_r(p'_r, \phi') = \int_{p'_r - \phi'}^{\infty} (v - p'_r) f(v) dv - \phi' F(p'_r - \phi') = \int_{p_w + c\tilde{w}}^{\infty} (v - p_w - c\tilde{w}) f(v) dv, \quad (8)$$

where $\epsilon > 0$. Since $u_r(p_r, \phi)$ decreases in both p_r and ϕ , in order for (8) to be satisfied, $\phi' > \phi$.

Suppose that ϵ is sufficiently small, so that (p'_r, ϕ') satisfies $p'_r > \phi'$. Now, note that

$$\begin{aligned} \pi(p_w, p'_r, \phi') - \pi(p_w, p_r, \phi) &= \pi_w(p_w) + \pi_r(p'_r, \phi') - \pi_w(p_w) - \pi_r(p_r, \phi) \\ &= \pi_r(p'_r, \phi') - \pi_r(p_r, \phi) \\ &= p'_r \bar{F}(p'_r - \phi') + \phi' F(p'_r - \phi') - p_r \bar{F}(p_r - \phi) - \phi F(p_r - \phi). \end{aligned}$$

From (7) and (8),

$$\int_{p_r - \phi}^{\infty} (v - p_r) f(v) dv - \phi F(p_r - \phi) = \int_{p'_r - \phi'}^{\infty} (v - p'_r) f(v) dv - \phi' F(p'_r - \phi'),$$

and rearranging the above equality gives

$$p'_r \bar{F}(p'_r - \phi') + \phi' F(p'_r - \phi') - p_r \bar{F}(p_r - \phi) - \phi F(p_r - \phi) = \int_{p'_r - \phi'}^{p_r - \phi} v f(v) dv,$$

so that $\pi(p_w, p'_r, \phi') - \pi(p_w, p_r, \phi) = \int_{p'_r - \phi'}^{p_r - \phi} v f(v) dv > 0$, i.e., $\pi(p_w, p'_r, \phi') > \pi(p_w, p_r, \phi)$.

From the above argument, we see that when $p_r > \phi$, the restaurant can always increase its profit without violating the participation constraint by lowering the reservation price and increasing the fee by small amount. Therefore, $p_r > \phi$ is not optimal.

Claim3: For a fixed p_w , setting (p_r, ϕ) to be $p_r < \phi$ cannot be optimal for the restaurant.

\therefore Suppose (p_r, ϕ) and (p'_r, ϕ') satisfy $p_r < \phi$, $p'_r = p_r + \epsilon$,

$$u_r(p_r, \phi) = \int_{p_r - \phi}^{\infty} (v - p_r) f(v) dv - \phi F(p_r - \phi) = \int_{p_w + c\tilde{w}}^{\infty} (v - p_w - c\tilde{w}) f(v) dv, \quad (9)$$

$$u_r(p'_r, \phi') = \int_{p'_r - \phi'}^{\infty} (v - p'_r) f(v) dv - \phi' F(p'_r - \phi') = \int_{p_w + c\tilde{w}}^{\infty} (v - p_w - c\tilde{w}) f(v) dv, \quad (10)$$

where $\epsilon > 0$. Since $u_r(p_r, \phi)$ decreases in both p_r and ϕ , in order for (10) to be satisfied, $\phi' < \phi$. Suppose that ϵ is sufficiently small, so that (p'_r, ϕ') satisfies $p'_r < \phi'$. From (7) and (8),

$$\pi(p_w, p'_r, \phi') - \pi(p_w, p_r, \phi) = \int_{p'_r - \phi'}^{p_r - \phi} v f(v) dv > 0.$$

so that $\pi(p_w, p'_r, \phi') > \pi(p_w, p_r, \phi)$. From the above argument, we see that when $p_r < \phi$, the restaurant can always increase its profit without violating the participation constraint by increasing the reservation price and decreasing the fee by small amount. Therefore, $p_r < \phi$ is not optimal.

From Claim2 and Claim3, we have shown that neither $p_r > \phi$ nor $p_r < \phi$ can be optimal. Therefore, the optimal price vector (p_r^*, ϕ^*) satisfies $p_r^* = \phi^*$.

Characterizing the optimal (p_r^*, ϕ^*) : From Claim1, we know that given a walk-in price p_w , (p_r^*, ϕ^*) satisfies

$$\int_{p_r^* - \phi^*}^{\infty} (v - p_r^*) f(v) dv - \phi^* F(p_r^* - \phi^*) = \int_{p_w + c\tilde{w}}^{\infty} (v - p_w - \tilde{w}) f(v) dv \quad (\text{IR}).$$

Since $p_r^* = \phi^*$, substituting ϕ^* to p_r^* in the (IR) constraint gives

$$p_r^* = \int_0^{\infty} v f(v) dv - \int_{p_w + c\tilde{w}}^{\infty} (v - p_w - c\tilde{w}) f(v) dv.$$

Proof of Proposition 2 We show this by contradiction. Suppose the optimal α were chosen so that $\alpha < 1$, and that $p_w^* < p_r^*$. Then the profit under (p_w^*, p_r^*, ϕ^*) satisfies

$$\pi(p_w^*, p_r^*, \phi^*) < \alpha \mu p_r^* + (1 - \alpha) \mu p_w^* \leq \mu p_r^*$$

Also, since the pure reservation price is $\int_0^{\infty} v f(v) df$ as shown in Lemma 1, and $p_r^* < \int_0^{\infty} v f(v) df$, it is better off for the restaurant to operate under pure reservation system. Thus, $\alpha^* < 1$ is a contradiction. This shows that when the restaurant optimally chooses to operate as a hybrid system, i.e., $\alpha^* \in (0, 1)$, then the prices should satisfy $p_w^* \geq p_r^*$.

Proof of Proposition 3 Let $\lambda_w(\Lambda, p_w)$ denote the equilibrium arrival rate for pure walk-in system when the market size is Λ and the walk-in price set by the restaurant is p_w . Similarly, $\lambda_r(\Lambda, p_r, \phi)$ denotes the equilibrium arrival rate for pure reservation system as a function of potential

market size and the price set for the reservations.

(i) There exists $\tilde{\Lambda}'$ such that $\pi_w^*(\Lambda, \mu) \leq \pi_r^*(\Lambda, \mu)$, for all $\Lambda \leq \tilde{\Lambda}'$:

\therefore Suppose two restaurants with the same market size and the same capacity, (Λ, μ) , with one operating under pure walk-in system and the other under pure reservation system. Consider the case where the restaurant under pure walk-in system charges the optimal price for its customers, p_w^* , while the restaurant under pure reservation system charges the reservation price and no-show fee equal to $(p_w^*, 0)$. Suppose $\Lambda \leq \mu$. Then the equilibrium arrival rates for the two restaurants become

$$\begin{cases} \lambda_w(\Lambda, p_w^*) = \Lambda \bar{F}(p_w^* + c\tilde{w}), \\ \lambda_r(\Lambda, p_w^*, 0) = \Lambda \bar{F}(p_w^*), \end{cases}$$

where $\tilde{w} = w(\lambda_w(\Lambda, p_w^*), \mu)$. Then,

$$\pi_r(\Lambda, p_w^*, 0) = p_w^* \Lambda \bar{F}(p_w^*) \geq p_w^* \Lambda \bar{F}(p_w^* + c\tilde{w}) = \pi_w(\Lambda, p_w^*)$$

When $\Lambda < \mu$, the equilibrium arrival rates become

$$\begin{cases} \lambda_w(\Lambda, p_w^*) = \Lambda \bar{F}(p_w^* + c\tilde{w}), \\ \lambda_r(\Lambda, p_w^*, 0) = \mu \bar{F}(p_w^*), \end{cases}$$

and $\pi_r(\Lambda, p_w^*, 0) \geq \pi_w(\Lambda, p_w^*)$ when $\mu \bar{F}(p_w^*) \geq \Lambda \bar{F}(p_w^* + cw(\mu \bar{F}(p_w^*), \mu))$, which can be rewritten as

$$\Lambda \leq \mu \frac{\bar{F}(p_w^*)}{\bar{F}(p_w^* + cw(\mu \bar{F}(p_w^*), \mu))}.$$

Also, note that when optimal price and no-show fee, (p_r^*, ϕ^*) , are charged, the reservation profit is greater than $\pi_r(\Lambda, p_w^*, 0)$. Therefore, when $\tilde{\Lambda}' = \mu \bar{F}(p_w^*) / \bar{F}(p_w^* + cw(\mu \bar{F}(p_w^*), \mu))$,

$$\pi_r^*(\Lambda) = \pi_r(\Lambda, p_r^*, \phi^*) \geq \pi_r(\Lambda, p_w^*, 0) \geq \pi_w(\Lambda, p_w^*) = \pi_w^*(\Lambda),$$

for all $\Lambda \leq \tilde{\Lambda}'$.

(ii) There exists $\tilde{\Lambda}''$ such that $\pi_w^*(\Lambda, \mu) > \pi_r^*(\Lambda, \mu)$, for all $\Lambda > \tilde{\Lambda}''$:

\therefore Let $\Lambda' = \Lambda + \epsilon$, with $\epsilon > 0$. Let $p_w^*(\Lambda)$ and $(p_r^*(\Lambda), \phi^*(\Lambda))$ denote the optimal price under pure walk-in system and that under pure reservation system, respectively, as a function of potential market size, Λ . We first consider the pure walk-in system. Define function $g_\Lambda(x)$ and $h_\Lambda(x)$ as

$$g_\Lambda(x) = \frac{x}{\Lambda},$$

$$h_{\Lambda}(x) = \bar{F}(p_w + cw(x, \mu)).$$

Note that $g_{\Lambda}(x)$ strictly increases in x with a range $[0, 1]$, while $h_{\Lambda}(x)$ decreases in x with range $[0, \bar{F}(p_w)]$, so the two curves intersect at a unique point. The equilibrium walk-in arrival rate, $\lambda_w(\Lambda, p_w)$, under the market size, Λ , and the price charged, p_w , is then given as the value, x_0 , where the two functions $g_{\Lambda}(x)$ and $h_{\Lambda}(x)$ meet, i.e., $g_{\Lambda}(x_0) = h_{\Lambda}(x_0)$. Now consider the functions

$$g_{\Lambda'}(x) = \frac{x}{\Lambda'},$$

$$h_{\Lambda'}(x) = \bar{F}(p_w + cw(x, \mu)).$$

Again, the equilibrium arrival rate under the market size Λ' and the price charged p_w is given as the point, x_1 , where $g_{\Lambda'}(x_1) = h_{\Lambda'}(x_1)$. Note that

$$g_{\Lambda'}(x_0) < g_{\Lambda}(x_0) = h_{\Lambda}(x_0) = h_{\Lambda'}(x_0).$$

Thus, the new equilibrium arrival rate under market size Λ' and price charged, p_w , should satisfy $x_1 > x_0$. From this fact we can see that

$$\lambda_w(\Lambda', p_w^*(\Lambda)) > \lambda_w(\Lambda, p_w^*(\Lambda)).$$

The optimal profit under market size Λ' , $\pi_w^*(\Lambda')$, then satisfies

$$\pi_w^*(\Lambda') = \pi_w(\Lambda', p_w^*(\Lambda')) \geq \pi_w(\Lambda', p_w^*(\Lambda)) = p_w^*(\Lambda) \lambda_w(\Lambda') > p_w^*(\Lambda) \lambda_w(\Lambda) = \pi_w(\Lambda, p_w^*(\Lambda)) = \pi_w^*(\Lambda).$$

The first inequality comes from optimality of $p_w^*(\Lambda')$, the following equality is the definition of $\pi_w(\Lambda', p_w^*(\Lambda))$, and the next inequality is what we just showed. Therefore, we have proven

$$\pi_w(\Lambda', p_w^*(\Lambda')) > \pi_w(\Lambda, p_w^*(\Lambda))$$

for $\Lambda' > \Lambda$. This implies that the optimal walk-in profit increases as Λ increases.

On the other hand, recall that the optimal price for pure reservation system is given as

$$p_r^*(\Lambda) = \phi^*(\Lambda) = \int_0^{\infty} v f(v) dv,$$

which is independent of Λ . Also, since the restaurant only takes reservation as much as its capacity accommodates, the total amount of reservation taken is restricted to μ and independent of Λ , when $\mu > \Lambda$. Thus, the optimal equilibrium reservation profit under market size Λ , $\pi_r^*(\Lambda)$, does not depend on Λ when Λ is sufficiently large.

From the above argument we know that for sufficiently large Λ , $\pi_w^*(\Lambda)$ goes up, while $\pi_r^*(\Lambda)$ remains constant as Λ increases. Therefore, we conclude that there exists $\tilde{\Lambda}''$ where $\pi_w^*(\Lambda) > \pi_r^*(\Lambda)$ for all $\Lambda > \tilde{\Lambda}''$.

From (ii), we know that the optimal profit for pure walk-in system, $\pi_w^*(\Lambda)$, increases in Λ , while the optimal profit for reservation system $\pi_r^*(\Lambda)$ remains constant over Λ when $\Lambda > \mu$. Also, we know that for all $\Lambda \leq \tilde{\Lambda}'$, $\pi_r^*(\Lambda) \geq \pi_w^*(\Lambda)$. Since $\tilde{\Lambda}' \geq \mu$, $\pi_r^*(\Lambda)$ remains constant for $\Lambda > \tilde{\Lambda}'$. Therefore, there exists a unique point $\tilde{\Lambda} \in [\tilde{\Lambda}', \tilde{\Lambda}'']$, where $\pi_r^*(\tilde{\Lambda}) = \pi_w^*(\tilde{\Lambda})$. For $\Lambda < \tilde{\Lambda}$, $\pi_r^*(\Lambda) > \pi_w^*(\Lambda)$, and for $\Lambda > \tilde{\Lambda}$, $\pi_r^*(\Lambda) < \pi_w^*(\Lambda)$.

Proof of Proposition 4 Assume $n > 1$, and let $\pi_w^*(\Lambda, \mu)$ and $\pi_r^*(\Lambda, \mu)$ denote the optimal profit under walk-in system and that under reservation system, respectively, when the market size is Λ and the capacity is μ . Also let $p_w^*(\Lambda, \mu)$ and $p_r^*(\Lambda, \mu)$ denote the optimal price for customers under walk-in system and that under reservation system, respectively, when the market size is Λ and the capacity is μ .

Claim1: $\pi_r^*(n\Lambda, n\mu) = n\pi_r^*(\Lambda, \mu)$.

\therefore Recall that $p_r^*(\Lambda, \mu) = \int_0^\infty v f(v) dv$ neither depends on Λ nor μ , so that $p_r^*(\Lambda, \mu) = p_r^*(n\Lambda, n\mu) = p_r^*$ for any n . Then we can write the optimal profits as

$$\pi_r^*(\Lambda, \mu) = \begin{cases} \Lambda p_r^*, & \text{if } \Lambda < \mu \\ \mu p_r^*, & \text{if } \Lambda \geq \mu, \end{cases}$$

and

$$\pi_r^*(n\Lambda, n\mu) = \begin{cases} n\Lambda p_r^*, & \text{if } \Lambda < \mu \\ n\mu p_r^*, & \text{if } \Lambda \geq \mu. \end{cases}$$

Therefore, $\pi_r^*(n\Lambda, n\mu) = n\pi_r^*(\Lambda, \mu)$.

Claim2: $\pi_w^*(n\Lambda, n\mu) > n\pi_w^*(\Lambda, \mu)$.

\therefore Note that from the general properties of queues, for a given arrival rate, λ , and a service rate, μ , $w(\lambda, \mu) < w(n\lambda, n\mu)$. Define $g_{\Lambda, \mu}(x)$ and $h_{\Lambda, \mu}(x)$ as

$$g_{\Lambda, \mu}(x) = \frac{x}{\Lambda},$$

$$h_{\Lambda, \mu}(x) = \bar{F}(p_w + cw(x, \mu)).$$

Note that $g_{\Lambda, \mu}(x)$ is increasing in x , whereas $h_{\Lambda, \mu}(x)$ is decreasing in x so that there exists a unique x where the two curves meet. Then for fixed Λ and μ , and given that the price charged by the restaurant is p_w , the equilibrium arrival rate $\lambda_w(\Lambda, \mu, p_w)$ is the value, x_0 where $g_{\Lambda, \mu}(x_0) = h_{\Lambda, \mu}(x_0)$. Now consider

$$g_{n\Lambda, n\mu}(x) = \frac{x}{n\Lambda},$$

$$h_{n\Lambda, n\mu}(x) = \bar{F}(p_w + cw(x, n\mu)),$$

and note that the equilibrium arrival rate, $\lambda_w(n\Lambda, n\mu, p_w)$, for market size, $n\Lambda$, and service rate, μ , with price charged, p_w , is given as the value, x_1 , with $g_{n\Lambda, n\mu}(x_1) = h_{n\Lambda, n\mu}(x_1)$. Now, note that

$$g_{n\Lambda, n\mu}(nx_0) = \frac{nx_0}{n\Lambda} = \frac{x_0}{\Lambda} = g_{\Lambda, \mu}(x_0),$$

while

$$h_{n\Lambda, n\mu}(nx_0) = \bar{F}(p_w + cw(nx_0, n\mu)) > \bar{F}(p_w + cw(x_0, \mu)) = h_{\Lambda, \mu}(x_0)$$

since $w(\lambda, \mu) < w(n\lambda, n\mu)$. Then we see that

$$g_{n\Lambda, n\mu}(nx_0) < h_{n\Lambda, n\mu}(nx_0),$$

therefore, the equilibrium point, x_1 , should satisfy $x_1 > nx_0$. This leads us to conclude that

$$\lambda_w(n\Lambda, n\mu, p_w^*(\Lambda, \mu)) > n\lambda_w(\Lambda, \mu, p_w^*(\Lambda, \mu)).$$

Then the optimal profit, $\pi_w^*(n\Lambda, n\mu)$ satisfies

$$\pi_w^*(n\Lambda, n\mu) = p_w^*(n\Lambda, n\mu) \cdot \lambda_w(n\Lambda, n\mu, p_w^*(n\Lambda, n\mu))$$

$$\begin{aligned}
&\geq p_w^*(\Lambda, \mu) \cdot \lambda_w(n\Lambda, n\mu, p_w^*(\Lambda, \mu)) \\
&> np_w^*(\Lambda, \mu) \cdot \lambda_w(\Lambda, \mu, p_w^*(\Lambda, \mu)) = n\pi_w^*(\Lambda, \mu),
\end{aligned}$$

where $p_w^*(\Lambda, \mu)$ represents the optimal price charged when the market size is Λ and the capacity is μ . The first equality is the definition of the optimal profit, the inequality after comes from the optimality of $p_w^*(n\Lambda, n\mu)$, the next inequality is what we just showed, and the last equality is again the definition of the optimal profit. Hereby, we have shown that

$$\pi_w^*(n\Lambda, n\mu) > n\pi_w^*(\Lambda, \mu).$$

Proof of Proposition 5 Let $\Lambda' > \Lambda$. Let (p_r, p_w) denote the optimal reservation price and the optimal walk-in price when the market size is Λ . Let (p'_r, p'_w) denote the reservation price and the walk-in price when the market size is Λ' and $\lambda_w(p'_w, \Lambda') = \lambda_w(p_w, \Lambda)$. Then p'_w should satisfy

$$(\Lambda' - \alpha\mu)\bar{F}(p'_w + c\tilde{w}) = \lambda_w(p'_w, \Lambda') = \lambda_w(p_w, \Lambda) = (\Lambda - \alpha\mu)\bar{F}(p_w + c\tilde{w}),$$

so that $p'_w > p_w$. Now note that

$$\begin{aligned}
p'_r - p_r &= \int_{p_w + c\tilde{w}}^{p'_w + c\tilde{w}} v f(v) dv + (p'_w + c\tilde{w})\bar{F}(p'_w + c\tilde{w}) - (p_w + c\tilde{w})\bar{F}(p_w + c\tilde{w}) \\
&= \int_{p_w + c\tilde{w}}^{p'_w + c\tilde{w}} v f(v) dv + (p'_w - p_w) - \{(p'_w + c\tilde{w})F(p'_w + c\tilde{w}) - (p_w + c\tilde{w})F(p_w + c\tilde{w})\} \\
&= (p'_w - p_w) - \int_{p_w + c\tilde{w}}^{p'_w + c\tilde{w}} F(v) dv \\
&< p'_w - p_w.
\end{aligned}$$

Thus, we see that profit from walk-in customers increases more than that from reservation customers, so that

$$\alpha^*(\Lambda') \leq \alpha^*(\Lambda).$$

Also note that $p_r \leq \int_0^\infty v f(v) dv$, while p_w can increase indefinitely as the market size increases.

Therefore, we can infer that

$$\lim_{\Lambda \rightarrow \infty} \alpha^*(\Lambda) = 0.$$

Proof of Proposition 6 We will first show the optimal price and fee under each strategy.

(i) When $E_\varepsilon[u_r^H] \geq E_\varepsilon[u_w^H]$, the proof is equivalent to the proof of Proposition 1.

(ii) When $E_\varepsilon[u_r^L] \geq E_\varepsilon[u_w^L]$, we know from the proof of Proposition 1 that the IC constraint should bind on the optimum point, (p_r^*, ϕ^*) , i.e., $E_\varepsilon[u_r^L(p_r^*, \phi^*)] = E_\varepsilon[u_w^L(p_r^*, \phi^*)]$, and the IC constraint is written by:

$$\int_{p_r - \phi - v_0^L}^{\infty} (v_0^L + \varepsilon)g(\varepsilon)d\varepsilon - p_r \bar{G}(p_r - \phi - v_0^L) - \phi^h G(p_r - \phi - v_0^L) = \int_{p_w + c\bar{w} - v_0^L}^{\infty} (v_0^L + \varepsilon - p_w - c\bar{w})g(\varepsilon)d\varepsilon$$

By implicit function theorem, we can compute

$$\left. \frac{d\phi}{dp_r} \right|_{E_\varepsilon[u_r^L] = E_\varepsilon[u_w^L]} = - \frac{\bar{G}(p_r - \phi - v_0^L)}{G(p_r - \phi - v_0^L)}.$$

Note that the profit function is given as

$$\begin{aligned} \pi(p_r, \phi, p_w) &= \mu q \{ p_r \bar{G}(p_r - \phi - v_0^H) + \phi G(p_r - \phi - v_0^H) \} \\ &\quad + \mu(1 - q) \{ p_r \bar{G}(p_r - \phi - v_0^L) + \phi G(p_r - \phi - v_0^L) \}, \end{aligned}$$

and

$$\begin{aligned} \frac{d\pi}{dp_r}(p_r, \phi, p_w) &= \mu q \left\{ \bar{G}(p_r - \phi - v_0^H) + \frac{d\phi}{dp_r} G(p_r - \phi - v_0^H) - (p_r - \phi)g(p_r - \phi - v_0^H) \left(1 - \frac{dp_r}{d\phi} \right) \right\} \\ &\quad + \mu(1 - q) \left\{ \bar{G}(p_r - \phi - v_0^L) + \frac{d\phi}{dp_r} G(p_r - \phi - v_0^L) - (p_r - \phi)g(p_r - \phi - v_0^L) \left(1 - \frac{dp_r}{d\phi} \right) \right\}. \end{aligned}$$

Then given $E_\varepsilon[u_r^L] = E_\varepsilon[u_w^L]$,

$$\begin{aligned} \frac{d\pi}{dp_r} &= \mu q \left\{ \bar{G}(p_r - \phi - v_0^H) - \frac{\bar{G}(p_r - \phi - v_0^L)}{G(p_r - \phi - v_0^L)} G(p_r - \phi - v_0^H) - (p_r - \phi)g(p_r - \phi - v_0^H) \left(1 - \frac{dp_r}{d\phi} \right) \right\} \\ &\quad + \mu(1 - q) \left\{ -(p_r - \phi)g(p_r - \phi - v_0^L) \left(1 - \frac{dp_r}{d\phi} \right) \right\}, \end{aligned}$$

so that when $p_r \leq \phi$,

$$\left. \frac{d\pi}{dp_r} \right|_{E_\varepsilon[u_r^L] = E_\varepsilon[u_w^L]} > 0.$$

Therefore, when $p_r \leq \phi$, $\pi(p_r, \phi, p_w)$ increases as increasing p_r and decreasing ϕ while keeping the IC constraint binding. Therefore, there is an incentive to deviate from (p_r, ϕ) such that $p_r \leq \phi$ and it is not an equilibrium. Hence, the equilibrium (p_r^*, ϕ^*) should satisfy $p_r^* > \phi^*$.

Now let (p_r^H, ϕ^H) and (p_r^L, ϕ^L) denote the optimal price vectors under scenario (i) and scenario (ii), respectively. Then

$$\pi(p_r^H, \phi^H) = \mu q \{p_r^H \bar{G}(p_r^H - \phi^H - v_0^H) + \phi^H G(p_r^H - \phi^H - v_0^H)\},$$

and

$$\begin{aligned} \pi(p_r^L, \phi^L) = & \mu q \{p_r^L \bar{G}(p_r^L - \phi^L - v_0^H) + \phi^L G(p_r^L - \phi^L - v_0^H)\} \\ & + \mu(1-q) \{p_r^L \bar{G}(p_r^L - \phi^L - v_0^L) + \phi^L G(p_r^L - \phi^L - v_0^L)\}. \end{aligned}$$

Then

$$\begin{aligned} \pi(p_r^H, \phi^H) - \pi(p_r^L, \phi^L) = & \mu q \left[\{p_r^H \bar{G}(p_r^H - \phi^H - v_0^H) + \phi^H G(p_r^H - \phi^H - v_0^H)\} \right. \\ & - \{p_r^L \bar{G}(p_r^L - \phi^L - v_0^H) + \phi^L G(p_r^L - \phi^L - v_0^H)\} \\ & \left. + \{p_r^L \bar{G}(p_r^L - \phi^L - v_0^L) + \phi^L G(p_r^L - \phi^L - v_0^L)\} \right] \\ & - \{p_r^L \bar{G}(p_r^L - \phi^L - v_0^L) + \phi^L G(p_r^L - \phi^L - v_0^L)\} \end{aligned}$$

Since

$$p_r^H \bar{G}(p_r^H - \phi^H - v_0^H) + \phi^H G(p_r^H - \phi^H - v_0^H) > p_r^L \bar{G}(p_r^L - \phi^L - v_0^H) + \phi^L G(p_r^L - \phi^L - v_0^H),$$

$\pi(p_r^H, \phi^H) - \pi(p_r^L, \phi^L)$ is linearly increasing in q , with $\pi(p_r^H, \phi^H) - \pi(p_r^L, \phi^L) < 0$ when $q = 0$ and $\pi(p_r^H, \phi^H) - \pi(p_r^L, \phi^L) > 0$ when $q = 1$. Thus, we can conclude that there exists $\tilde{q} \in (0, 1)$ such that

$$\begin{cases} \pi(p_r^H, \phi^H) \geq \pi(p_r^L, \phi^L) & \text{if } q \geq \tilde{q}, \\ \pi(p_r^H, \phi^H) < \pi(p_r^L, \phi^L) & \text{if } q < \tilde{q}. \end{cases}$$

Proof of Proposition 7 For now, we do not assume any condition on the value of Λ and μ , and show general properties of the optimal price vector for both cases shown in the proposition.

Claim1: $\int_{p_r^{\circ*} - \phi^{\circ*}}^{\infty} (v - p_r^{\circ}) f(v) dv - \phi^{\circ} F(p_r^{\circ} - \phi^{\circ}) = 0$:

We show this by contradiction. Suppose that $(p_r^{\circ}, p_r^{\circ}, \phi^{\circ})$ are given so that $\int_{p_r^{\circ*} - \phi^{\circ*}}^{\infty} (v - p_r^{\circ}) f(v) dv - \phi^{\circ} F(p_r^{\circ} - \phi^{\circ}) > 0$. Then

(i) When $\Lambda \geq \mu/\bar{F}(p_r^{\circ} - \phi^{\circ})$: The reservation profit is

$$\pi_r^{\circ}(p_r^{\circ}, \phi^{\circ}) = \mu/\bar{F}(p_r^{\circ} - \phi^{\circ}) \{p_r^{\circ} \bar{F}(p_r^{\circ} - \phi^{\circ}) + \phi^{\circ} F(p_r^{\circ} - \phi^{\circ})\},$$

and

$$\frac{d}{dp_r^o} \pi_r^o = \frac{d}{dp_r^o} \left\{ \mu p_r^o + \mu \phi^o \frac{F(p_r^o - \phi^o)}{\bar{F}(p_r^o - \phi^o)} \right\} > 0.$$

Therefore, the restaurant can increase p_r^o without violating the (IR) constraint, and increase its profit.

(ii) When $\Lambda < \mu/\bar{F}(p_r^o - \phi^o)$: The profit for the restaurant is given as

$$\pi_r^o(p_r^o, \phi^o) = \Lambda \{ p_r^o \bar{F}(p_r^o - \phi^o) + \phi^o F(p_r^o - \phi^o) \},$$

so that

$$\begin{aligned} \frac{d}{dp_r^o} \pi_r^o &= \Lambda \{ \bar{F}(p_r^o - \phi^o) + (p_r^o - \phi^o) f(p_r^o - \phi^o) \} > 0, \quad \text{if } p_r^o < \phi^o, \\ \frac{d}{d\phi^o} \pi_r^o &= \Lambda \{ (p_r^o - \phi^o) f(p_r^o - \phi^o) + F(p_r^o - \phi^o) \} > 0, \quad \text{if } p_r^o > \phi^o. \end{aligned}$$

Therefore, again the restaurant can either increase p_r^o or ϕ^o and earn more profit without violating the (IR) constraint. From (i) and (ii), we have shown that the optimal price vector (p_r^{o*}, ϕ^{o*}) should satisfy $\int_{p_r^{o*} - \phi^{o*}}^{\infty} (v - p_r^o) f(v) dv - \phi^o F(p_r^o - \phi^o) = 0$.

Claim2: $p_r^o < \phi^o$ is not optimal, i.e., $p_r^{o*} \geq \phi^{o*}$. Suppose (p_r^o, ϕ^o) satisfies

$$p_r^o < \phi^o \quad \text{and} \quad \int_{p_r^o - \phi^o}^{\infty} (v - p_r^o) f(v) dv - \phi^o F(p_r^o - \phi^o) = 0.$$

(i) When $\Lambda \geq \mu/\bar{F}(p_r^o - \phi^o)$: The profit maximization problem for the restaurant becomes

$$\begin{aligned} \max_{p_r^o, \phi^o} \quad & \Lambda \{ p_r^o \bar{F}(p_r^o - \phi^o) + \phi^o F(p_r^o - \phi^o) \} \\ \text{s.t.} \quad & \int_{p_r^o - \phi^o}^{\infty} (v - p_r^o) f(v) dv - \phi^o \bar{F}(p_r^o - \phi^o) \geq 0, \end{aligned}$$

which is equivalent to the profit maximization problem without overbooking. With the same reasoning as in the proof of Proposition 1, increasing p_r^o and decreasing ϕ^o , while keeping the (IR) constraint binding, increases the restaurant's profit. Therefore, $p_r^o < \phi^o$ is not optimal.

(ii) When $\Lambda < \mu/\bar{F}(p_r^o - \phi^o)$: The profit for the restaurant is given as

$$\pi_r^o(p_r^o, \phi^o) = \frac{\mu}{\bar{F}(p_r^o - \phi^o)} \{ p_r^o \bar{F}(p_r^o - \phi^o) + \phi^o F(p_r^o - \phi^o) \}.$$

Suppose (p_r^o, ϕ^o) , and $(p_r^{o'}, \phi^{o'})$ satisfy $p_r^o < p_r^{o'} < \phi^{o'} < \phi^o$, and

$$\int_{p_r^o - \phi^o}^{\infty} (v - p_r^o) f(v) dv - \phi^o \bar{F}(p_r^o - \phi^o) = 0 \quad (11)$$

$$\int_{p_r^{o'} - \phi^{o'}}^{\infty} (v - p_r^{o'}) f(v) dv - \phi^{o'} \bar{F}(p_r^{o'} - \phi^{o'}) = 0 \quad (12)$$

Then from (11) and (12),

$$\begin{aligned} & p_r^{o'} \bar{F}(p_r^{o'} - \phi^{o'}) + \phi^{o'} F(p_r^{o'} - \phi^{o'}) - p_r^o \bar{F}(p_r^o - \phi^o) - \phi^o F(p_r^o - \phi^o) \\ &= \int_{p_r^{o'} - \phi^{o'}}^{\infty} v f(v) dv - \int_{p_r^o - \phi^o}^{\infty} v f(v) dv = \int_{p_r^o - \phi^o}^{p_r^{o'} - \phi^{o'}} v f(v) dv > 0. \end{aligned}$$

Now,

$$\begin{aligned} & \pi_r^o(p_r^{o'} - r, \phi^{o'}) - \pi_r^o(p_r^o - r, \phi^o) \\ &= \frac{\mu}{\bar{F}(p_r^{o'} - \phi^{o'})} \left\{ p_r^{o'} \bar{F}(p_r^{o'} - \phi^{o'}) + \phi^{o'} F(p_r^{o'} - \phi^{o'}) \right\} - \frac{\mu}{\bar{F}(p_r^o - \phi^o)} \left\{ p_r^o \bar{F}(p_r^o - \phi^o) + \phi^o F(p_r^o - \phi^o) \right\} \\ &\geq \frac{\mu}{\bar{F}(p_r^o - \phi^o)} \left\{ p_r^{o'} \bar{F}(p_r^{o'} - \phi^{o'}) + \phi^{o'} F(p_r^{o'} - \phi^{o'}) - p_r^o \bar{F}(p_r^o - \phi^o) - \phi^o F(p_r^o - \phi^o) \right\} \\ &> 0. \end{aligned}$$

Thus, when $p_r^o < \phi^o$, the restaurant can increase its revenue without violating the (IR) constraint by increasing p_r^o and decreasing ϕ^o . Therefore, $p_r^o < \phi^o$ is not optimal for the restaurant.

Claim3: Proposition 7.(i) If $\Lambda \leq \mu/\bar{F}(0)$, $p_r^{o*} = \phi^{o*} = \int_0^{\infty} v f(v) dv$.

From Claim2, we know that $p_r^o < \phi^o$ is never optimal, so that for all pairs of (p_r^o, ϕ^o) that are candidates for the optimal price, $(p_r^{o*} < \phi^{o*})$, satisfy $p_r^o \geq \phi^o$. Suppose $\Lambda \leq \mu/\bar{F}(0)$. Then, for all (p_r^o, ϕ^o) that satisfies $p_r^o \geq \phi^o$,

$$\frac{\mu}{\bar{F}(p_r^o - \phi^o)} \geq \frac{\mu}{\bar{F}(0)} \geq \Lambda.$$

In this cases, the restaurant takes $\Lambda = \min \{ \Lambda, \mu/\bar{F}(p_r^o - \phi^o) \}$ reservations and the profit maximization problem for the restaurant becomes

$$\max_{p_r^o, \phi^o} \Lambda \left\{ p_r^o \bar{F}(p_r^o - \phi^o) + \phi^o F(p_r^o - \phi^o) \right\} \quad (13)$$

$$s.t. \quad \int_{p_r^o - \phi^o}^{\infty} (v - p_r^o) f(v) dv - \phi^o \bar{F}(p_r^o - \phi^o) = 0 \quad (14)$$

$$p_r^o \geq \phi^o \quad (15)$$

Then, similarly as in Proposition 1, we can prove that

$$p_r^{o*} = \phi^{o*} = \int_0^\infty v f(v) dv$$

when $\Lambda \leq \mu/\bar{F}(0)$.

Claim4: Proposition 7.(ii) If $\Lambda > \mu/\bar{F}(0)$, $\mu = \Lambda\bar{F}(p_r^{o*} - \phi^{o*})$.

(a) Suppose $p_r^o - \phi^o > 0$ and $\Lambda \geq \mu/\bar{F}(p_r^o - \phi^o)$. Then the profit function can be rewritten as

$$\begin{aligned} \pi_r^o(p_r^o, \phi^o) &= \frac{\mu}{\bar{F}(p_r^o - \phi^o)} \{p_r^o \bar{F}(p_r^o - \phi^o) + \phi^o F(p_r^o - \phi^o)\} \\ &= \frac{\mu}{\bar{F}(p_r^o - \phi^o)} \int_{p_r^o - \phi^o}^\infty v f(v) dv. \end{aligned}$$

Let $\xi = p_r^o - \phi^o$. Then

$$\frac{d}{d\xi} = \frac{\mu f(\xi)}{\{\bar{F}(\xi)\}^2} \left\{ \int_\xi^\infty v f(v) dv - \xi \bar{F}(\xi) \right\} > 0,$$

so that the profit increases as the gap, $p_r^o - \phi^o$, widens.

(b) Now suppose $p_r^o - \phi^o > 0$ and $\Lambda < \mu/\bar{F}(p_r^o - \phi^o)$. Then the profit function becomes

$$\pi_r^o(p_r^o, \phi^o) = \Lambda \{p_r^o \bar{F}(p_r^o - \phi^o) + \phi^o F(p_r^o - \phi^o)\}$$

and the profit maximization problem become identical to (13). We know, from the proof of Proposition 4, that in this case, the profit increases as the gap, $p_r^o - \phi^o$, becomes smaller.

From (a) and (b) we have shown that the profit function is unimodal in the gap between the price and the fee, $p_r^o - \phi^o$, and the maximum is obtained at (p_r^{o*}, ϕ^{o*}) where $\mu = \Lambda\bar{F}(p_r^{o*} - \phi^{o*})$.

Claim5: Proposition 7.(ii) If $\Lambda > \mu/\bar{F}(0)$, $p_r^{o*} > \phi^{o*}$, $p_r^{o*} > p_r^*$, $\phi^{o*} < \phi^*$.

We have already shown that $p_r^{o*} > \phi^{o*}$ by proving $\mu = \Lambda\bar{F}(p_r^{o*} - \phi^{o*})$. Since the optimal overbooking price vector, (p_r^{o*}, ϕ^{o*}) , satisfies the same (IR) constraint as the optimal price vector without overbooking, (p_r^*, ϕ^*) , in order for $p_r^{o*} > \phi^{o*}$ to be satisfied, $p_r^{o*} > p_r^*$ and $\phi^{o*} < \phi^*$.